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Fractional calculus of modified special functions involving the generalized M-series in their kernels and illustrative examples

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ABSTRACT

In this paper we apply the Riemann–Liouville, Erdelyi–Kober and Caputo fractional operators to modified beta, Gauss hypergeometric and confluent hypergeometric functions in which the generalized M-series are included in their kernels. Furthermore, as examples, we obtain solutions of some fractional differential equations involving the above modified special functions.

1. Introduction and preliminaries

The concept of fractional calculus was first introduced in 1695 through correspondence between Leibniz and L'Hospital, as evidenced by sources such as Refs. 1–4. The letters discuss the formula

$$D^n f(x) = \frac{d^n f(x)}{dx^n}, \quad \text{for} \quad n > 0$$

and the question of whether the number n can be extended to fractional numbers. The question's answer resulted in the creation of a new theory by scientists known as fractional calculus. Numerous studies have been conducted on fractional calculus, as seen in Refs. 5–12.

The Riemann–Liouville fractional integral (RLFI)[1] of order $\varepsilon \in \mathbb{C}$ is given by

$$\left[I_{0+}^{\varepsilon} f \right](\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1} f(\omega) d\omega,$$

$$(\rho > 0, \Re(\varepsilon) > 0).$$

The Erdelyi–Kober fractional integral (EKFI)[1] for $\sigma > 0$, $\mu \in \mathbb{C}$ is defined by

$$\left[I_{0+;\sigma,\mu}^{\varepsilon} f \right](\rho) = \frac{\sigma \rho^{-\sigma(\varepsilon+\mu)}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho^{\sigma}-\omega^{\sigma})^{\varepsilon-1} \omega^{\sigma\mu+\sigma-1} f(\omega) d\omega, \tag{1}$$

$$(\rho > 0, \Re(\varepsilon) > 0).$$

In this paper, we will use the following Erdelyi–Kober fractional integral (EKFI) for $\sigma = 1$ in Eq. (1):

$$\left[I_{0+;\mu}^{\varepsilon} f \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1} \omega^{\mu} f(\omega) d\omega,$$

$$(\rho > 0, \Re(\varepsilon) > 0).$$

The Riemann–Liouville fractional derivative (RLFD)[1] of order $\varepsilon \in \mathbb{C}$ for $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is given by

$$\left[D_{0+}^{\varepsilon} f \right](\rho) = \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{d\rho^m} \int_0^{\rho} (\rho-\omega)^{m-\varepsilon-1} f(\omega) d\omega,$$

$$(\rho > 0, \Re(\varepsilon) > 0).$$

The Caputo fractional derivative (CFD)[1] for $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is defined by

$$\left[{}^{c}D_{0+}^{\varepsilon} f \right](\rho) = \frac{1}{\Gamma(m-\varepsilon)} \int_0^{\rho} (\rho-\omega)^{m-\varepsilon-1} f^{(m)}(\omega) d\omega,$$

$$(\rho > 0, \Re(\varepsilon) > 0).$$

Special functions have been in the focus of attention of scientists due to their popularity in disciplines such as mathematics, physics, engineering. Many scientists have published quite a lot of studies in

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recent years on various generalizations of special functions, see for example Refs. 13–33 and the reference therein.

One of the popular special function is ${}_p^{\alpha}M_q^{\beta}$ which defined for $\Re(\alpha) > 0$ as

$$ {}_p^{\alpha}M_q^{\beta}(z) = {}_p^{\alpha}M_q^{\beta}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;z) = \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta)}, \quad (2) $$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;j=1,\dots,q)$. If $p\leq q$ then (2) is convergent for all z. If $p=q+1$ it is also convergent for $|z|<\delta=\alpha^{\alpha}$, but if $p>q+1$ it is divergent. If $p=q+1$, (2) can be convergent for $|z|=\delta$ depending on the conditions of the parameters[34]. ${}_p^{\alpha}M_q^{\beta}$ is also known as the generalized M-series.

Remark 1. Here, it is important to note that the generalized M-series is a special case of the Wright generalized hypergeometric function ${}_p\Psi_q(z)$; it is easy to find its relationship with various classical special and trigonometric functions that is why it is more general in nature. It is also important because its basic cases are followed by the Mittag-Leffler function and the hypergeometric functions, and all these functions have indeed found key applications in solving problems in applied sciences, chemistry, physics and biology.

The symbol $(\cdot)_n$ used to denote the Pochhammer symbol[35], which defined by

$$ (\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)} = \begin{cases} \zeta(\zeta+1)\dots(\zeta+n-1), & n=1,2,\dots, \\ 1, & n=0. \end{cases} \quad (3) $$

The modified gamma and beta functions for $\Re(\rho)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\alpha)>0$ was introduced by Ata in Ref. 16, respectively, as follows:

$$ {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(x;\rho) = {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;x;\rho) = \int_0^{\infty} \Delta^{x-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;-\Delta-\frac{\rho}{\Delta}\right)d\Delta $$

and

$$ {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) = {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;x,y;\rho) = \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta, $$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;j=1,\dots,q)$.

In the same paper the author used the modified beta function to defined the modified Gauss and confluent hypergeometric functions for $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$, respectively, as follows:

$$ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {}^{M}F_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\chi_1,\chi_2;\chi_3;z;\rho) = \sum_{n=0}^{\infty}(\chi_1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}, \quad (|z|<1) $$

and

$$ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\chi_2;\chi_3;z;\rho) = \sum_{n=0}^{\infty} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}, $$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;j=1,\dots,q)$.

If we take $\rho=0$ and $p=q=\xi_1=\eta_1=\alpha=\beta=1$ in the modified special functions given above, we get the classic forms of corresponding special functions[35].

It should be noted that the relationship between the classic gamma and beta functions, as described in Ref. 35, is as follows:

$$ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (\Re(x)>0,\Re(y)>0). \quad (4) $$

The paper proposes the use of modified special functions involving generalized M-series, which contain more parameters. It is expected that this will expand the application areas.

The remainder of this paper is planned as follows: In Section 2, we apply Riemann–Liouville, Caputo and Erdelyi–Kober fractional operators to M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions. In Section 3, we obtain solutions of fractional differential equations including M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions.

2. Fractional calculus of modified special functions

In this section, we apply Riemann–Liouville fractional integral, Erdelyi–Kober fractional integral, Riemann–Liouville fractional derivative, and Caputo fractional derivative operators to M-beta, M-Gauss hypergeometric, and M-confluent hypergeometric functions.

2.1. Riemann–Liouville fractional integral (RLFI)

Theorem 2.1. *Let $\Re(\varepsilon)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\alpha)>0$. Then, applying the RLFI to the M-beta function yields the following formula:*

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;x,y;\rho)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}, $$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;j=1,\dots,q)$.

Proof. Applying the RLFI to the M-beta function, we have

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{1}{\Gamma(\varepsilon)}\int_0^{\rho}(\rho-\omega)^{\varepsilon-1}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\omega)d\omega. $$

Using the definition of the M-beta function, we get

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{1}{\Gamma(\varepsilon)}\int_0^{\rho}(\rho-\omega)^{\varepsilon-1}\int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\frac{-\omega}{\Delta(1-\Delta)}\right)d\Delta d\omega. $$

Using the definition of the generalized M-series, we obtain

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{1}{\Gamma(\varepsilon)}\int_0^{\rho}(\rho-\omega)^{\varepsilon-1}\int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n}\frac{\left(\frac{-\omega}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}d\Delta d\omega. $$

Taking $\omega=\rho u$ and using Eq. (4), we have

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \rho^{\varepsilon}\int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}\frac{\Gamma(n+1)}{\Gamma(n+1+\varepsilon)}d\Delta. $$

Multiplying by $\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\varepsilon)}$ and using Eq. (3), we get

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{\rho^{\varepsilon}}{\Gamma(1+\varepsilon)}\int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\dots(\xi_p)_n(1)_n}{(\eta_1)_n\dots(\eta_q)_n(1+\varepsilon)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}d\Delta. $$

Considering the definition of the generalized M-series, we obtain

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{\rho^{\varepsilon}}{\Gamma(1+\varepsilon)}\int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_{p+1}^{\ \alpha}M_{q+1}^{\beta}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta. $$

Thus, we have the desired formula as:

$$ \left[I_{0+}^{\varepsilon}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho)\right](\rho) = \frac{{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;x,y;\rho)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}, $$

which completes the proof. □

Theorem 2.2. *Let $\Re(\varepsilon) > 0$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$. Then, applying the RLFI to the M-Gauss hypergeometric function yields the following formula:*

$$\left[I^{\varepsilon}_{0+} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}F^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}},$$

where $\arg(1-z) < \pi$ *and* $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;\,j=1,\dots,q)$.

Proof. Applying the RLFI to the M-Gauss hypergeometric function, we have

$$\left[I^{\varepsilon}_{0+} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega = \rho u$ and using Eq. (4), we get

$$\left[I^{\varepsilon}_{0+} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{\varepsilon}}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1)}{\Gamma(n+1+\varepsilon)} d\Delta.$$

Multiplying by $\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\varepsilon)}$ and using Eq. (3), we obtain

$$\left[I^{\varepsilon}_{0+} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{\varepsilon}}{\Gamma(1+\varepsilon)B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times {}_{p+1}^{\alpha}M^{\beta}_{q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[I^{\varepsilon}_{0+} \, {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}F^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}},$$

which completes the proof. □

Theorem 2.3. *Let $\Re(\varepsilon) > 0$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$. Then, applying the RLFI to the M-confluent hypergeometric function yields the following formula:*

$$\left[I^{\varepsilon}_{0+} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}},$$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;\,j=1,\dots,q)$.

Proof. Applying the RLFI to the M-confluent hypergeometric function, we have

$$\left[I^{\varepsilon}_{0+} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega = \rho u$ and using Eq. (4), we get

$$\left[I^{\varepsilon}_{0+} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{\varepsilon}}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1)}{\Gamma(n+1+\varepsilon)} d\Delta.$$

Multiplying by $\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\varepsilon)}$ and using Eq. (3), we obtain

$$\left[I^{\varepsilon}_{0+} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{\varepsilon}}{\Gamma(1+\varepsilon)B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times {}_{p+1}^{\alpha}M^{\beta}_{q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[I^{\varepsilon}_{0+} \, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}},$$

which completes the proof. □

2.2. Erdelyi–Kober fractional integral (EKFI)

Theorem 2.4. *Let $\Re(\varepsilon) > 0$, $\Re(1+\mu) > 0$, $\Re(1+\mu+\varepsilon) > 0$, $\Re(x) > 0$, $\Re(y) > 0$, $\Re(\alpha) > 0$. Then, applying the EKFI to the M-beta function yields the following formula:*

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\, {}^{M}B^{(\alpha,\beta)}_{p+1,q+1}(\xi_1,\dots,\xi_p,1+\mu;\eta_1,\dots,\eta_q,1+\mu+\varepsilon;x,y;\rho)}{\Gamma(1+\mu+\varepsilon)},$$

where $\xi_i,\eta_j \neq 0,-1,-2,\dots\,(i=1,\dots,p;\,j=1,\dots,q)$.

Proof. Applying the EKFI to the M-beta function, we have

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\omega^{\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\omega)d\omega.$$

Using the definition of the M-beta function, we get

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\omega^{\mu} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_{p}^{\alpha}M^{\beta}_{q}\left(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\frac{-\omega}{\Delta(1-\Delta)}\right) d\Delta d\omega.$$

Using the definition of the generalized M-series, we obtain

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\omega^{\mu} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n} \frac{\left(\frac{-\omega}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} d\Delta d\omega.$$

Taking $\omega = \rho u$ and using Eq. (4), we have

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n}{(\eta_1)_n\dots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1+\mu)}{\Gamma(n+1+\mu+\varepsilon)} d\Delta.$$

Multiplying by $\frac{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}$ and using Eq. (3), we get

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\varepsilon)} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\dots(\xi_p)_n(1+\mu)_n}{(\eta_1)_n\dots(\eta_q)_n(1+\mu+\varepsilon)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} d\Delta.$$

Considering the definition of the generalized M-series, we obtain

$$\left[I^{\varepsilon}_{0+;\mu} \, {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\varepsilon)} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_{p+1}^{\alpha}M^{\beta}_{q+1}\left(\xi_1,\dots,\xi_p,1+\mu;\eta_1,\dots,\eta_q,1+\mu+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\ {}^{M}B^{(\alpha,\beta)}_{p+1,q+1}(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;x,y;\rho)}{\Gamma(1+\mu+\varepsilon)},$$

which completes the proof. □

Theorem 2.5. *Let $\Re(\varepsilon)>0$, $\Re(1+\mu)>0$, $\Re(1+\mu+\varepsilon)>0$, $\Re(\alpha)>0$, $\Re(\chi_3)>\Re(\chi_2)>0$. Then, applying the EKFI to the M-Gauss hypergeometric function yields the following formula:*

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\ {}^{M}F^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\mu+\varepsilon)},$$

where $\arg(1-z)<\pi$ *and* $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Applying the EKFI to the M-Gauss hypergeometric function, we have

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\omega^{\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega=\rho u$ and using Eq. (4), we get

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1+\mu)}{\Gamma(n+1+\mu+\varepsilon)} d\Delta.$$

Multiplying by $\frac{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}$ and using Eq. (3), we obtain

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\varepsilon)B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times {}_{p+1}^{\ \alpha}M^{\beta}_{q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}F^{(\alpha,\beta)}_{p,q}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\ {}^{M}F^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\mu+\varepsilon)},$$

which completes the proof. □

Theorem 2.6. *Let $\Re(\varepsilon)>0$, $\Re(1+\mu)>0$, $\Re(1+\mu+\varepsilon)>0$, $\Re(\alpha)>0$, $\Re(\chi_3)>\Re(\chi_2)>0$. Then, applying the EKFI to the M-confluent hypergeometric function yields the following formula:*

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\ {}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\mu+\varepsilon)},$$

where $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Applying the EKFI to the M-confluent hypergeometric function, we have

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon-\mu}}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\omega^{\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega=\rho u$ and using Eq. (4), we get

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1+\mu)}{\Gamma(n+1+\mu+\varepsilon)} d\Delta.$$

Multiplying by $\frac{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}{\Gamma(1+\mu)\Gamma(1+\mu+\varepsilon)}$ and using Eq. (3), we obtain

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\varepsilon)B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times {}_{p+1}^{\ \alpha}M^{\beta}_{q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[I^{\varepsilon}_{0+;\mu}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\Gamma(1+\mu)\ {}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1+\mu;\eta_1,\ldots,\eta_q,1+\mu+\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1+\mu+\varepsilon)},$$

which completes the proof. □

2.3. Riemann–Liouville fractional derivative (RLFD)

Theorem 2.7. *Let $0<\Re(\varepsilon)<1$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\alpha)>0$. Then, applying the RLFD to the M-beta function yields the following formula:*

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{{}^{M}B^{(\alpha,\beta)}_{p+1,q+1}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;x,y;\rho)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Applying the RLFD for $m=1$ to the M-beta function, we have

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{1}{\Gamma(1-\varepsilon)} \frac{d}{d\rho} \int_0^{\rho} (\rho-\omega)^{-\varepsilon}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\omega)d\omega.$$

Using the definition of the M-beta function, we get

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{1}{\Gamma(1-\varepsilon)} \frac{d}{d\rho} \int_0^{\rho} (\rho-\omega)^{-\varepsilon} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_{p}^{\alpha}M^{\beta}_{q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\frac{-\omega}{\Delta(1-\Delta)}\right) d\Delta d\omega.$$

Using the definition of the generalized M-series, we obtain

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{1}{\Gamma(1-\varepsilon)} \frac{d}{d\rho} \int_0^{\rho} (\rho-\omega)^{-\varepsilon} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n} \frac{\left(\frac{-\omega}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} d\Delta d\omega.$$

Taking $\omega=\rho u$ and using Eq. (4), we have

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} \frac{\Gamma(n+1)\Gamma(1-\varepsilon)}{\Gamma(n+1-\varepsilon)} d\Delta.$$

Using Eq. (3), we get

$$\left[D^{\varepsilon}_{0+}\ {}^{M}B^{(\alpha,\beta)}_{p,q}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times \sum_{n=0}^{\infty} \frac{(\xi_1)_n\cdots(\xi_p)_n(1)_n}{(\eta_1)_n\cdots(\eta_q)_n(1-\varepsilon)_n} \frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)} d\Delta.$$

Considering the definition of the generalized M-series, we obtain

$$\left[D_{0+}^{\varepsilon} \, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} \times {}_{p+1}^{\alpha}M_{q+1}^{\beta}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we have the desired formula as:

$$\left[D_{0+}^{\varepsilon} \, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) \right](\rho) = \frac{{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;x,y;\rho)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

which completes the proof. □

Theorem 2.8. *Let* $0<\Re(\varepsilon)<1$, $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$. *Then, applying the RLFD to the M-Gauss hypergeometric function yields the following formula:*

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}F_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\arg(1-z)<\pi$ *and* $\xi_i,\eta_j\neq 0,-1,-2,\ldots\,(i=1,\ldots,p;\,j=1,\ldots,q)$.

Proof. Applying the RLFD for $m=1$ to the M-Gauss hypergeometric function, we have

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{\Gamma(1-\varepsilon)}\frac{d}{d\rho}\int_0^{\rho}(\rho-\omega)^{-\varepsilon}\, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega=\rho u$ and using Eq. (4), we get

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}\frac{\Gamma(n+1)\Gamma(1-\varepsilon)}{\Gamma(n+1-\varepsilon)}d\Delta.$$

Using Eq. (3), we obtain

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\cdots(\xi_p)_n(1)_n}{(\eta_1)_n\cdots(\eta_q)_n(1-\varepsilon)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}d\Delta.$$

Considering the definition of the generalized M-series, we have

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \times {}_{p+1}^{\alpha}M_{q+1}^{\beta}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we get the desired formula as:

$$\left[D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}F_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

which completes the proof. □

Theorem 2.9. *Let* $0<\Re(\varepsilon)<1$, $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$. *Then, applying the RLFD to the M-confluent hypergeometric function yields the following formula:*

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}\Phi_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\xi_i,\eta_j\neq 0,-1,-2,\ldots\,(i=1,\ldots,p;\,j=1,\ldots,q)$.

Proof. Applying the RLFD for $m=1$ to the M-confluent hypergeometric function, we have

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{1}{\Gamma(1-\varepsilon)}\frac{d}{d\rho}\int_0^{\rho}(\rho-\omega)^{-\varepsilon}\, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\omega)d\omega.$$

Taking $\omega=\rho u$ and using Eq. (4), we get

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\cdots(\xi_p)_n}{(\eta_1)_n\cdots(\eta_q)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}\frac{\Gamma(n+1)\Gamma(1-\varepsilon)}{\Gamma(n+1-\varepsilon)}d\Delta.$$

Using Eq. (3), we obtain

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times \sum_{n=0}^{\infty}\frac{(\xi_1)_n\cdots(\xi_p)_n(1)_n}{(\eta_1)_n\cdots(\eta_q)_n(1-\varepsilon)_n}\frac{\left(\frac{-\rho}{\Delta(1-\Delta)}\right)^n}{\Gamma(\alpha n+\beta)}d\Delta.$$

Considering the definition of the generalized M-series, we have

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{\rho^{-\varepsilon}}{\Gamma(1-\varepsilon)B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta) \times {}_{p+1}^{\alpha}M_{q+1}^{\beta}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

Thus, we get the desired formula as:

$$\left[D_{0+}^{\varepsilon} \, {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}\Phi_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

which completes the proof. □

2.4. Caputo fractional derivative (CFD)

Theorem 2.10. *Let* $0<\Re(\varepsilon)<1$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\alpha)>0$. *Then, applying the CFD to the M-beta function yields the following formula:*

$$\left[{}^{c}D_{0+}^{\varepsilon} \, {}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) \right](\rho) = \frac{{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;x,y;\rho)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\xi_i,\eta_j\neq 0,-1,-2,\ldots\,(i=1,\ldots,p;\,j=1,\ldots,q)$.

Proof. Applying the CFD for $m=1$ to the M-beta function and performing the required calculations, the proof is complete. □

Theorem 2.11. *Let* $0<\Re(\varepsilon)<1$, $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$. *Then, applying the CFD to the M-Gauss hypergeometric function yields the following formula:*

$$\left[{}^{c}D_{0+}^{\varepsilon} \, {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) \right](\rho) = \frac{{}^{M}F_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_1,\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\arg(1-z)<\pi$ *and* $\xi_i,\eta_j\neq 0,-1,-2,\ldots\,(i=1,\ldots,p;\,j=1,\ldots,q)$.

Proof. Applying the CFD for $m=1$ to the M-Gauss hypergeometric function and performing the required calculations, the proof is complete. □

Theorem 2.12. *Let $0 < \Re(\varepsilon) < 1$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$. Then, applying the CFD to the M-confluent hypergeometric function yields the following formula:*

$$\left[\, {}^{c}D^{\varepsilon}_{0+}\, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}(\chi_2;\chi_3;z;\rho) \,\right](\rho) = \frac{{}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1-\varepsilon;\chi_2;\chi_3;z;\rho\right)}{\Gamma(1-\varepsilon)\rho^{\varepsilon}},$$

where $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Applying the CFD for $m = 1$ to the M-confluent hypergeometric function and performing the required calculations, the proof is complete. □

3. Illustrative examples

Let $\Re(\varepsilon) > 0$ and $m-1 < \Re(\varepsilon) < m$ for $m \in \mathbb{N}$. If $1 \leqq p \leqq \infty$ and $f \in I^{\varepsilon}_{a+}(L_p)$, then Ref. 1:

$$\left[\, I^{\varepsilon}_{a+} D^{\varepsilon}_{a+} f \,\right](\rho) = f(\rho), \tag{5}$$

holds almost everywhere on $[a,b]$ and where $I^{\varepsilon}_{a+}(L_p)$ as follows[1]:

$$I^{\varepsilon}_{a+}(L_p) = \left\{ f : f(\rho) = \left[\, I^{\varepsilon}_{a+}\varphi \,\right](\rho),\ \varphi(\rho) \in L_p(a,b) \right\}.$$

In this section, we obtain solutions of fractional differential equations involving the modified beta, Gauss hypergeometric and confluent hypergeometric functions using Eq. (5).

Example 1. Let $1 < \Re(\varepsilon) < 2$, $\Re(x) > 0$, $\Re(y) > 0$, $\Re(\alpha) > 0$. We consider the fractional differential equation

$$\left[\, D^{\varepsilon}_{0+} f \,\right](\rho) = {}^{M}B^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;x,y;\frac{\rho}{\varepsilon}\right), \tag{6}$$

where $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Considering Eq. (5) for $a = 0$ and applying the RLFI to the fractional differential equation, we have

$$\left[\, I^{\varepsilon}_{0+} D^{\varepsilon}_{0+} f \,\right](\rho) = \left[\, I^{\varepsilon}_{0+}\, {}^{M}B^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;x,y;\frac{\rho}{\varepsilon}\right) \right](\rho).$$

That is,

$$f(\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\ {}^{M}B^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;x,y;\frac{\omega}{\varepsilon}\right) d\omega.$$

Taking $\omega = \rho u$ and performing the necessary calculations, we get the solution function as:

$$f(\rho) = \frac{{}^{M}B^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\varepsilon;x,y;\frac{\rho}{\varepsilon}\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}. \quad □$$

Corollary 3.1. *If we use the generalized beta function defined by Chaudhry et al.[20] instead of the modified beta function in Eq. (6), we obtain the solution function as follows:*

$$f(\rho) = \rho^{\varepsilon} \sum_{n=0}^{\infty} \frac{\Gamma(x-n)\Gamma(y-n)}{\Gamma(n+1+\varepsilon)\Gamma(x+y-2n)} \left(-\frac{\rho}{\varepsilon}\right)^n.$$

Example 2. Let $1 < \Re(\varepsilon) < 2$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$. We consider the fractional differential equation

$$\left[\, D^{\varepsilon}_{0+} f \,\right](\rho) = {}^{M}F^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_1,\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right), \tag{7}$$

where $\arg(1-z) < \pi$ and $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Considering Eq. (5) for $a = 0$ and applying the RLFI to the fractional differential equation, we have

$$\left[\, I^{\varepsilon}_{0+} D^{\varepsilon}_{0+} f \,\right](\rho) = \left[\, I^{\varepsilon}_{0+}\, {}^{M}F^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_1,\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right) \right](\rho).$$

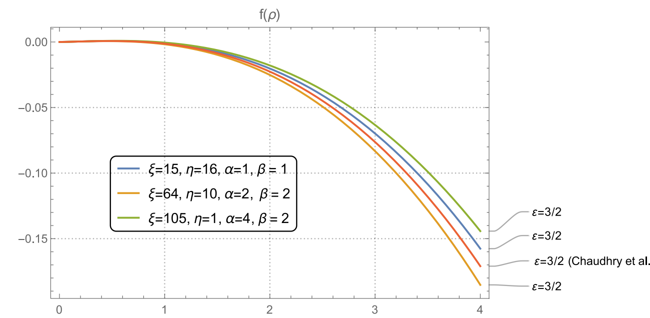

Fig. 1. The behavior of approximate solutions $f(\rho)$ of Example 1 and Corollary 3.1 for $\varepsilon = 3/2$, where $p = q = 1$, $x = y = 4$, $0 < \rho < 4$ and generalized M-series and exponential series indexes $n = 0, 1$.

That is,

$$f(\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\ {}^{M}F^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_1,\chi_2;\chi_3;z;\frac{\omega}{\varepsilon}\right) d\omega.$$

Taking $\omega = \rho u$ and performing the necessary calculations, we get the solution function as:

$$f(\rho) = \frac{{}^{M}F^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\varepsilon;\chi_1,\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}. \quad □$$

Corollary 3.2. *If we take the generalized Gauss hypergeometric function defined by Chaudhry et al.[21] instead of the modified Gauss hypergeometric function in Eq. (7), we obtain the solution function as follows:*

$$f(\rho) = \frac{\Gamma(\chi_3)\rho^{\varepsilon}}{\Gamma(\chi_1)\Gamma(\chi_2)\Gamma(\chi_3-\chi_2)} \times \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(\chi_1+k)\Gamma(\chi_2+k-n)\Gamma(\chi_3-\chi_2-n)}{\Gamma(1+n+\varepsilon)\Gamma(\chi_3+k-2n)} \left(-\frac{\rho}{\varepsilon}\right)^n \frac{z^k}{k!}.$$

Example 3. Let $1 < \Re(\varepsilon) < 2$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$. We consider the fractional differential equation

$$\left[\, D^{\varepsilon}_{\rho} f \,\right](\rho) = {}^{M}\Phi^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right), \tag{8}$$

where $\xi_i,\eta_j \neq 0,-1,-2,\ldots\ (i=1,\ldots,p;\ j=1,\ldots,q)$.

Proof. Considering Eq. (5) for $a = 0$ and applying the RLFI to the fractional differential equation, we have

$$\left[\, I^{\varepsilon}_{0+} D^{\varepsilon}_{0+} f \,\right](\rho) = \left[\, I^{\varepsilon}_{0+}\, {}^{M}\Phi^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right) \right](\rho).$$

That is,

$$f(\rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^{\rho} (\rho-\omega)^{\varepsilon-1}\ {}^{M}\Phi^{(\alpha,\beta)}_{p,q}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_2;\chi_3;z;\frac{\omega}{\varepsilon}\right) d\omega.$$

Taking $\omega = \rho u$ and performing the necessary calculations, we get the solution function as:

$$f(\rho) = \frac{{}^{M}\Phi^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\varepsilon;\chi_2;\chi_3;z;\frac{\rho}{\varepsilon}\right)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}. \quad □$$

Corollary 3.3. *If we take the generalized confluent hypergeometric function defined by Chaudhry et al.[21] instead of the modified confluent hypergeometric function in Eq. (8), we obtain the solution function as follows:*

$$f(\rho) = \frac{\Gamma(\chi_3)\rho^{\varepsilon}}{\Gamma(\chi_2)\Gamma(\chi_3-\chi_2)} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(\chi_2+k-n)\Gamma(\chi_3-\chi_2-n)}{\Gamma(1+n+\varepsilon)\Gamma(\chi_3+k-2n)} \left(-\frac{\rho}{\varepsilon}\right)^n \frac{z^k}{k!}.$$

4. Conclusion

In this paper, we applied RLFI, EKFI, RLFD and CFD to M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions. We

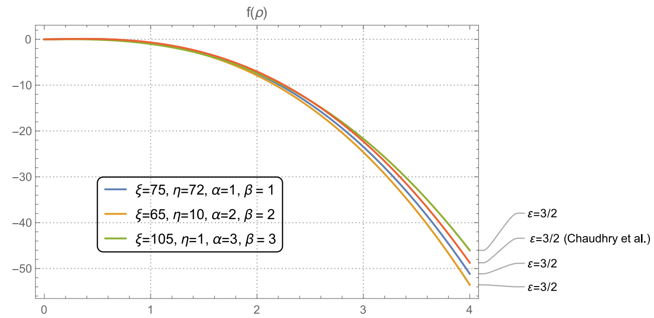

Fig. 2. The behavior of approximate solutions $f(\rho)$ of Example 2 and Corollary 3.2 for $\varepsilon = 3/2$, where $p = q = 1$, $\chi_1 = \chi_2 = 2$, $\chi_3 = 4$, $z = 1/2$, $0 < \rho < 4$ and generalized M-series and exponential series indexes $n = 0, 1$ and modified Gauss hypergeometric series and generalized Gauss hypergeometric series indexes $k = 0, 1$.

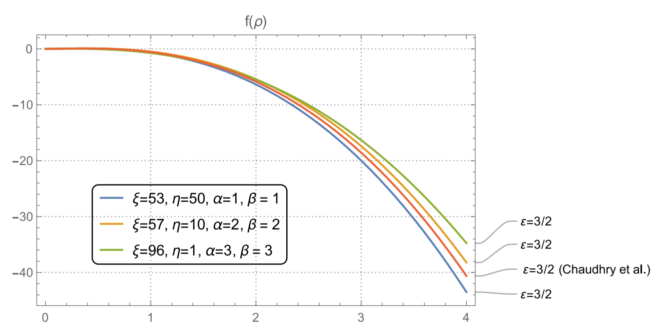

Fig. 3. The behavior of approximate solutions $f(\rho)$ of Example 3 and Corollary 3.3 for $\varepsilon = 3/2$, where $p = q = 1$, $\chi_2 = 2$, $\chi_3 = 4$, $z = 1/2$, $0 < \rho < 4$ and generalized M-series and exponential series indexes $n = 0, 1$, modified confluent hypergeometric series and generalized confluent hypergeometric series indexes $k = 0, 1$.

would also like to point out that the results of the RLFD and CFD applied to the M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions overlap. Then, we obtained solutions of fractional differential equations involving the M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions.

In Fig. 1, the approximate behavior of the solutions of fractional differential equations involving the modified beta function and the generalized beta function defined by Chaudhry et al.[20] is compared.

In Fig. 2, the approximate behavior of the solutions of fractional differential equations involving the modified Gauss hypergeometric function and the generalized Gauss hypergeometric function defined by Chaudhry et al.[21] is compared.

In Fig. 3, the approximate behavior of the solutions of fractional differential equations involving the modified confluent hypergeometric function and the generalized confluent hypergeometric function defined by Chaudhry et al.[21] is compared.

Note that in Figs. 1, 2, and 3, the red line represents the approximate behavior of the solutions of fractional differential equations involving the generalized special functions defined by Chaudhry et al. and the blue, yellow and green lines represent the approximate behavior of the solutions of fractional differential equations involving the modified special functions.

Declaration of competing interest

The author declares that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

Data availability

No data was used for the research described in the article.

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